TULSA GIRLS' MATH CIRCLE

ZOME TOOL at The Tulsa Girls' Math Circle

April, 2016

Zometools consist of balls and struts. The struts come in three colors: blue, yellow and red. Each color of strut comes in three lengths: short, medium, long. We use subscripts to indicate size. Thus b_1 is a short blue and r_3 is a long red.

Build the following:

a) An equilateral triangle with blue sides
b) A b_1 , b_1, b_2 triangle
c) A b_2, b_2, b_1 triangle
d) A y_1, y_1, b_1 triangle
e) A r_1, r_1, b_1 triangle
f) A square
g) A regular pentagon
h) A regular hexagon

The length of a strut is not its physical length. Instead, we put balls on each end and define its length to be the distance between the centers of these two balls. This is not a weird definition; it is the only sensible definition, since zome objects are built with struts AND balls.

1. Using this definition, verify that

$$x_1 + x_2 = x_3$$

For any color *b, r,* or *y.*

2. Try making "medium" and "large" versions of some of the triangles that you built earlier. Use similar triangles to conclude that

$$\frac{b_3}{b_2} = \frac{b_2}{b_1}$$

And

$$\frac{y_2}{y_1} = \frac{r_2}{r_1} = \frac{b_2}{b_1}$$

And

$$\frac{y_3}{y_2} = \frac{r_3}{r_2} = \frac{b_3}{b_2}$$

Call this common ratio, τ .

3. Prove that $\tau^2 = \tau + 1$.
4. Show that τ is the famous *golden ratio* :

$$\tau = \frac{1+\sqrt{5}}{2}$$

5. We could also do the following: build a *b_3, b_3, b_2* isosceles triangle. Replace one of the *b_3* sides with a *b_2* and a *b_1* and then use another *b_2* to make the famous similar triangles inside the *golden triangle.*

6. Fun with τ: the formula $\tau^2 = \tau + 1$ means that any polynomial in τ can be reduced to a linear expression in τ. Practice with this: show that $\tau^3 = 2\tau + 1$ and $1/_{\tau} = \tau - 1$. Find a simple expression for τ^7. See anything familiar?
7. *Duality of Dodecahedron and Icosahedron:*
   a. Build a "starburst" by using one Zomeball and filling every triangular hole with *y_2* strut lollipops (a *y_2* strut with a Zomeball on one end) , and every hexagonal hole with *r_1* strut lollipops.
   b. Connect the Zomeballs at the ends of the red struts with *b_1* struts. What polyhedron does this represent? How do the yellow struts relate to this polyhedral?
   c. Connect the Zomeballs at the ends of the yellow struts with *b_1* struts. What polyhedron does this represent? How are the vertices of the previous polyhedron related to this polyhedron?

d. Let's find a formula for the volume of the dodecahedron with side length 1. Build a dodecahedon with *b_1* struts.
   a. Discover a cube of side length b_2 hidden in a dodecahedron. Build this cube inside your dodecahedron. Be carefull! Then you can think of the dodecahedron as a cube plus 6 identical "roof" structures. Verify that the side length of the cube is τ. Thus, if we can compute the volume *R* of a roof structure, the volume of the entire dodecahedron will be *$\tau^3 + 6R$* .

   Here we go:
   b. Show that the height of the roof structure is $\frac{1}{2}$.
   c. The roof structure can be broken into three parts: a central triangular prism, with identical structures on either side of the prism which can be shoved together to form a pyramid with a rectangular base. Show that the volume of the prism is $^{\tau}/_{4}$.
   d. Show that the volume of this pyramid equals $^{1}/_{6}$, and that the base is not just any rectangle, but------a golden rectangle!
   e. Put it all together to show that the volume of the dodecahedron is:

$$\frac{8+7\tau}{2} = \frac{15+7\sqrt{5}}{4}$$

Taken from *Proof by Zomes*, Mathpath 2007, Paul Zeitz